

Gödel Number Encoding From Mathematics Into Arithmetic

written by Cheng Li | April 12, 2025



Kurt Gödel's Incompleteness Theorems revolutionized mathematical logic by demonstrating the inherent limitations of formal systems. A crucial component of his proof was the technique known as Gödel numbering, which allows mathematical statements, proofs, and logical operations to be encoded as numbers. This technique transformed logic into arithmetic, enabling self-referential statements that underpinned his groundbreaking theorems. This essay explores the concept of Gödel numbering, its applications, and its broader implications for mathematics and computer science.

Gödel numbering is a method that assigns a unique natural number to each symbol, formula, and sequence within a formal system. This encoding allows statements about logic to be transformed into statements about numbers, making it possible to discuss the properties of a formal system within itself.

To achieve this, each basic symbol (such as logical operators, variables, and quantifiers) is assigned a unique natural number. More complex structures, such as formulas and proofs, are then encoded using these basic numbers through specific encoding rules. The primary purpose of Gödel numbering is to enable self-referential statements in formal systems, ultimately proving the incompleteness of arithmetic.

Construction of Gödel Numbers

- Assigning Numbers to Symbols:** Each symbol in a formal system is given a unique number. For instance:
 - “ \forall ” (Universal quantifier) = 1
 - “ \exists ” (Existential quantifier) = 2
 - “+” (Addition operator) = 3
 - “=” (Equality symbol) = 4
 - “0” (Zero) = 5
 - “S” (Successor function) = 6
- Encoding Formulas:**
Encoding Formulas: Sequences of symbols are converted into a single number using a prime factorization technique. If a formula consists of symbols numbered a_1, a_2, a_3, \dots , its Gödel number is given by: $G(F) = 2^{a_1} \times 3^{a_2} \times 5^{a_3} \times \dots \times p_n^{a_n}$ where p_n is the n -th prime number.
- Encoding Proofs:** Since proofs are sequences of formulas, each proof can be encoded by assigning a unique number to each step and applying the same encoding technique.

By utilizing this method, every formula and proof in arithmetic can be uniquely represented by a natural number.

Examples of Gödel Numbering

To illustrate, consider a simple arithmetic statement: “ $0 + S(0) = S(0)$ ” (i.e., $0 + 1 = 1$).

- Suppose we assign the numbers:
 - “0” = 5
 - “S” = 6
 - “+” = 3
 - “=” = 4
- The expression “ $0 + S(0) = S(0)$ ” can be represented as a sequence of numbers: (5, 3, 6, 5, 4, 6, 5).
- Using the encoding function: $G(F) = 2^5 \times 3^3 \times 3^6 \times 7^5 \times 11^4 \times 13^6 \times 17^5$. This large number uniquely represents the formula within the system.

Gödel numbering bridges arithmetic and logic, enabling the formalization of self-referential statements and proving fundamental limits in mathematics. Beyond its role in Gödel's Incompleteness Theorems, it has influenced fields such as computer science, cryptography, and formal verification. The concept remains one of the most significant breakthroughs in logic, highlighting both the power and limitations of formal systems.